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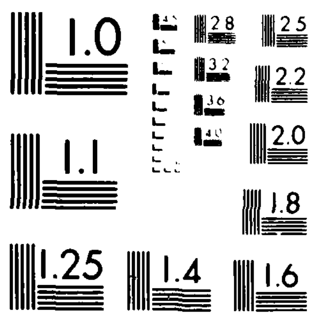


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Within the framework of an existing purely mechanical, rate-type theory of plasticity detailed calculations are presented for certain types of material response during stress and strain cycling in a uniaxial homogeneous deformation. These features pertain specifically to material response in stress cycling between fixed values of stress in tension and compression (not necessarily equal in magnitude) resulting in ratcheting of strain, and a type of saturation hardening caused by strain cycling between any (continued)

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20. Abstract (continued)

two fixed values of strain when the mean value of stress (in tension and compression) tends to zero.

# Calculations for Uniaxial Stress and Strain Cycling

# in Plasticity

by

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Abstract. Within the framework of an existing purely mechanical, rate-type theory of plasticity detailed calculations are presented for certain types of material response during stress and strain cycling in a uniaxial homogeneous deformation. These features pertain specifically to material response in stress cycling between fixed values of stress in tension and compression (not necessarily equal in magnitude) resulting in ratcheting of strain, and a type of saturation hardening caused by strain cycling between any two fixed values of strain when the mean value of stress (in tension and compression) tends to zero.

## 1. Introduction

Starting from a general theory of plasticity developed in stress space by Green and Naghdi [1,2] and its formulation relative to strain space by Naghdi and Trapp [3], Caulk and Naghdi [4] have discussed hardening response and in particular have compared the predictions of the theory with two sets of experimental results. In [4], with the limitation to small deformation, systematic specialization of the constitutive equations of the general theory were specifically applied to saturation hardening and other types of stress or strain cycling were not considered. Additional developments of saturation hardening in the context of strain-space formulation were discussed by Casey and Naghdi [5]; and, for the class of materials for which detailed comparisons with experiments were made in [4], the results for saturation hardening in [4] and [5] are equivalent.

More recently, the stress and strain cycling has also been discussed by Drucker and Palgen [6], who employed an approach to plasticity different from that of strain-space formulation utilized in [4,5]. In the course of their development, Drucker and Palgen [6] have also discussed the material response appropriate for unsymmetric plastic cycles of stress or of strain. While it may not be obvious at first sight, one would expect that a suitable specialization of the general theory should be capable of accommodating the type of unsymmetric cycles mentioned and discussed by Drucker and Palgen [6]. The purpose of this paper is to show that this is the case. In particular, in what follows we demonstrate the predictive capability of the theory in [1,2,3] for two types of uniaxial material response, namely

1) ratcheting of strain caused by stress cycling between fixed values of stress in tension and compression (not necessarily equal in magnitude) similar to that shown in Fig. 1 and

2) progressive relaxation to zero of the mean stress caused by strain cycling between any two fixed values of strain similar to that shown in Fig. 2.

The above two types of responses in the light of experimental results were described and discussed by Drucker and Palgen [6, p. 480]. By way of background and for definiteness, some features of these two types of responses are recalled here with reference to a uniaxial homogeneous deformation in a tension-compression test. In the context of small deformation, let e and s stand, respectively, for the component $e_{11}$ of the strain tensor and the component $s_{11}$ of the stress tensor*. Then, a stress cycle such as $A_1B_1C_1D_1E_1F_1A_2$ in Fig. 1 consists of loading in one direction from some chosen value of stress, say at $A_1$, to a limiting value of stress at $C_1$ followed by reversing the direction of loading until another limiting value of stress (not necessarily equal to that at $C_1$) is reached at $F_1$ and the direction of loading is again reversed until the stress reaches a value at $A_2$ equal to that of the initial choice at $A_1$. More generally, again with reference to Fig. 1, during the $n^{th}$ cycle $(n = 1,2,3,...)$ the response which begins at any point such as $A_n, B_n, ...,$ or $F_n$ and ends at a corresponding value of stress at $A_{n+1}, B_{n+1}, ...,$ or $F_{n+1}$ constitutes a stress cycle. Consider now a specimen cyclically loaded into the plastic range between stress limits whose average is nonzero§ i.e., stress limits which are unsymmetric about the strain axis in the s-e plane. During a typical stress cycle of this kind (e.g., $A_2B_2C_2D_2E_2F_2A_3$ in Fig. 1) the strain at the beginning and end of the cycle (and hence also the minimum and maximum values of strain attained during the cycle) vary from one cycle to the next. Evidently,

*A distinction between the Cauchy stress tensor or either of the Piola-Kirchhoff stresses is unnecessary when only a small deformation is being considered.

§This feature is referred to in [6] as "unsymmetric cycles of stress in the plastic range."

experimental observations (see, for example Dolan [7, Fig. 14]) indicate that if the average of the stress limits* is positive, both the minimum and maximum strains attained in a given cycle can progressively increase from one cycle to the next. Similarly, it may be inferred from the discussion in Sandor's book [8, pp. 97-100] that if the average of the stress limits is negative, then both the minimum and maximum values of strain attained in a given cycle can progressively decrease from one cycle to the next. Thus, while the stress limits remain fixed for repeated cyclic loading, the strain limits of the cycles can be progressively displaced along the strain axis from one cycle to the next**. A well-known phenomenon of this type, called ratcheting of strain, refers to progressive increase in plastic strain during cycling between any two fixed values of stress levels. Evidently such a progressive increase in plastic strain may take place irrespective of the starting point in the cycle. Indeed, as is clear from Fig. 1, the strains at $A_{n+1}, B_{n+1}, \ldots, F_{n+1}$ all have been increased along the e-axis from the values of strain at $A_n, B_n, \ldots, F_n$.

For cycling in the plastic range between fixed strain limits, on the other hand, for most metals the extent of the elastic range during each successive strain cycle increases (or decreases) until a limiting periodic response is reached in which the stress-strain curve of each succeeding cycle is the same (see Fig. 3). This phenomenon, which was previously discussed in [4], is sometimes called saturation hardening and a material is said to saturate when it reaches this limiting behavior. Moreover, if the

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*The terminology of the average of the stress level is referred to in [6] as the "mean stress" and "mean stress in the cycle."

**The strain limits of the cycles will be displaced in the positive strain direction if the average of the stress limits is positive (as depicted in Fig. 1) and in the negative strain direction if the average of the strain limits is negative (see [8, Fig. 98]).

specimen is cycled between fixed strain limits whose average is nonzero, it has been observed by Morrow and Sinclair [9] and by Dolan [7, Fig. 15][§] that for a certain class of materials the average of the limiting stresses in the cycle will progressively relax and tend to zero. For example, with reference to Fig. 2, the average value of the stresses at $C_n$ and $F_n$ $(n \geq 3)$ tends to zero for repeated cyclic loading between fixed strain limits. For completeness, we also briefly indicate in Fig. 3 the prediction of the theory (by means of the procedure of the present paper) for the case of saturation hardening, and examine its comparison with that in [4].

[§] See also [6, Fig. 2].

## 2. Basic Equations

Using a strain-space formulation of plasticity, we summarize here the basic equations of a purely mechanical theory contained in the papers of Green and Naghdi [1,2], Naghdi and Trapp [3] and Casey and Naghdi [5]. In addition to the strain tensor $e_{KL}$, we assume the existence of a symmetric second order plastic strain $e^p_{KL}$ and a scalar $\kappa$ representing a measure of work-hardening. It is assumed that the stress tensor $s_{KL}$ is given by a constitutive equation of the form

$$s_{KL} = \hat{s}_{KL}(\mathcal{U}) \quad , \quad \mathcal{U} = (e_{MN}, e^p_{MN}, \kappa) \tag{1}$$

and that, for fixed values of $e^p_{MN}$ and $\kappa$, $(1)_1$ can be inverted in the form

$$e_{KL} = \hat{e}_{KL}(\mathcal{V}) \quad , \quad \mathcal{V} = (s_{MN}, e^p_{MN}, \kappa) \quad . \tag{2}$$

The response functions $\hat{s}_{KL}$ and $\hat{e}_{KL}$ in (1) and (2) are taken to be smooth. We admit the existence of a smooth yield (or loading) function $g(\mathcal{U})$ in strain space such that for fixed values of $(e^p_{KL}, \kappa)$ the equation $g(\mathcal{U}) = 0$ represents a closed orientable hypersurface of dimension five enclosing a region of strain space. With the use of $(2)_1$, we obtain a corresponding function $f(\mathcal{V})$ through the formula

$$g(\mathcal{U}) = g(\hat{e}_{MN}(\mathcal{V}), e^p_{MN}, \kappa) = f(\mathcal{V}) \quad . \tag{3}$$

Because of the assumed smoothness of $(2)_1$, for fixed values of $(e^p_{KL}, \kappa)$ the equation $f(\mathcal{V}) = 0$ represents a hypersurface in stress space having the same geometrical properties as the hypersurface in strain space.

The loading criteria of strain-space formulation will be regarded as primary. Also, for our present purpose, it is sufficient to consider only a special case of the stress response in which $\hat{s}_{KL}$ in $(1)_1$ is independent

of $\kappa$ and its dependence on the variables $(e_{MN}, e^p_{MN})$ occurs only through the difference $e_{MN} - e^p_{MN}$. Then, the constitutive equations for the rate of plastic strain $\dot{e}^p_{KL}$ and rate of work-hardening parameter $\dot{\kappa}$ during loading (i.e., when $g = 0$ and $\hat{g} > 0$)* may be expressed as (see the development between Eqs. (36)-(42) in [5]):

$$\dot{e}^p_{KL} = \frac{\hat{g}}{\Gamma + \Lambda} \frac{\partial f}{\partial s_{KL}} \quad , \quad \dot{\kappa} = C_{KL} \dot{e}^p_{KL} \quad , \tag{4}$$

where

$$\hat{g} = \frac{\partial g}{\partial e_{KL}} \dot{e}_{KL} \quad , \tag{5}$$

$$\Lambda = \frac{\partial f}{\partial s_{KL}} \frac{\partial g}{\partial e_{KL}} \quad , \quad \Gamma = - \frac{\partial f}{\partial s_{KL}} \left( \frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} C_{KL} \right) \quad , \quad \Gamma + \Lambda > 0 \tag{6}$$

and where the response coefficients $C_{KL}$ are functions of the variables $(1)_2$. It should be noted that in obtaining the expression $(4)_1$ a condition resulting from a work inequality proposed in [10] has been also imposed on the constitutive equation for plastic strain rate.

The characterization of strain-hardening behavior by means of the rate-independent quotient $\hat{f}/\hat{g}$, with $\hat{f} \equiv (\partial f / \partial s_{KL}) \dot{s}_{KL}$, is discussed in [5]. More recently, a scalar quantity $\Phi$ is defined in [11] in such a way that does not presuppose a condition of loading and is such that during loading it has the same value as $\hat{f}/\hat{g}$. For our present purpose, the scalar function $\Phi$ can be defined as [12, Eq. (4.13)]

$$\Phi = \frac{\Gamma}{\Gamma + \Lambda} \quad , \tag{7}$$

and a particle of an elastic-plastic material is then said to exhibit [11,12]

*The scalar $\hat{g}$ is defined by Eq. (5) below.

$$
\begin{cases}
\text{hardening behavior if and only if } \Phi > 0 \quad , & \text{(a)} \\
\text{softening behavior if and only if } \Phi < 0 \quad , & \text{(b)} \\
\text{perfectly plastic behavior if and only if } \Phi = 0 \quad , & \text{(c)}
\end{cases}
\tag{8}
$$

Moreover, the strain-hardening response of a material is said to saturate to one of the values $\{K_h, K_s, 0\}$ and exhibits either (a) saturation hardening, (b) saturation softening and (c) saturation to perfectly plastic behavior, in that order, if

$$
\lim_{t \to \infty} \Phi = \begin{cases}
K_h > 0 \quad , & \text{(a)} \\
K_s < 0 \quad , & \text{(b)} \\
0 \quad . & \text{(c)}
\end{cases}
\tag{9}
$$

## 3. Special Constitutive Response Functions

It is convenient to prescribe the various constitutive response functions in terms of the components of deviatoric stress $\tau_{KL}$ and deviatoric strain $\gamma_{KL}$, namely

$$\begin{aligned} \tau_{KL} &= s_{KL} - \bar{s}\,\delta_{KL} \quad , \quad \tau_{KK} = 0 \quad , \\ \gamma_{KL} &= e_{KL} - \bar{e}\,\delta_{KL} \quad , \quad \gamma_{KK} = 0 \quad , \end{aligned} \tag{10}$$

with similar definition for the components of deviatoric plastic strain $\gamma^{p}_{KL}$, where $\bar{s}$ and $\bar{e}$ are the mean normal stress and mean normal strain, respectively. In the context of small deformation and for materials which are isotropic in the reference state, we specify the stress response by generalized Hooke's law and this can be written in the form

$$\tau_{KL} = 2\mu(\gamma_{KL} - \gamma^{p}_{KL}) \quad , \quad \bar{s} = 3k(\bar{e} - \bar{e}^{p}) \quad , \tag{11}$$

where $\mu$ is the shear modulus of elasticity and $k$ is the bulk modulus.

In regard to the characterization of material response to stress and strain cycling of the types referred to under 1) and 2) in Section 1, we first observe that it is the translation of the yield surface (whose interior represents the elastic domain), and not its size, which is particularly significant here. For the response described under 1) in Section 1, while the plastic strain progressively increases from one cycle to the next, the overall translation of the yield surface in stress space is not significantly different between the later and earlier cycles. For the response described under 2) in Section 1, on the other hand, the amount of translation of the yield surface in stress space becomes progressively smaller as the material is cycled between fixed strain limits. In the yield function utilized previously by Caulk and Naghdi [4, Eq. (40)], the quantity representing translation of the yield surface in stress space can be regarded

as a constant represented by $\frac{1}{2}\alpha$ times the plastic strain. Hence, at a given value of plastic strain the translation of the yield surface is determined independently of the loading history. By contrast, in our present discussion it is important to account for the effect of loading history pertinent to the translation of the yield surface. A simple way to achieve this is to generalize the restricted equations of Caulk and Naghdi [4, Eqs. (40), (56) and $(70)_1$] by replacing the constant coefficient $\alpha$ with a scalar function $\hat{\alpha}$ of the work-hardening parameter $\kappa$ which is $O(1)$. In particular we take $\hat{\alpha}$ to be linear in $\kappa$, with an initial value $\alpha_o$ when $\kappa = \kappa_o$ and a value $\alpha_s$ at saturation when $\kappa = \kappa_s$. Thus, we specify the loading function in the form

$$f = (\tau_{KL} - \frac{\hat{\alpha}(\kappa)}{2}\gamma^p_{KL})(\tau_{KL} - \frac{\hat{\alpha}(\kappa)}{2}\gamma^p_{KL}) - \kappa$$

$$= 4\mu^2[\gamma_{KL} - (1 + \frac{\hat{\alpha}(\kappa)}{4\mu})\gamma^p_{KL}][\gamma_{KL} - (1 + \frac{\hat{\alpha}(\kappa)}{4\mu})\gamma^p_{KL}] - \kappa = g \quad , \tag{12}$$

with $\hat{\alpha}(\kappa)$ given by

$$\hat{\alpha}(\kappa) = \frac{(\alpha_o - \alpha_s)\kappa + (\alpha_s\kappa_o - \alpha_o\kappa_s)}{\kappa_o - \kappa_s} \quad . \tag{13}$$

It should be noted that for the special case in which $\alpha_o = \alpha_s = \alpha$, $\hat{\alpha}(\kappa)$ becomes simply $\alpha$ and the loading functions (12) reduce to those used by Caulk and Naghdi [4, Eqs. $(40)_1$, (56), $(70)_1$]. In (12), $\frac{1}{2}\hat{\alpha}(\kappa)\gamma^p_{KL}$ may be interpreted as the center of the yield surface in the deviatoric stress space. Similarly, the quantity $(1 + \frac{1}{4\mu}\hat{\alpha}(\kappa))\gamma^p_{KL}$ may be interpreted as the center of the yield surface in deviatoric strain space. The size of the loading surfaces (which represents the extent of the elastic region) depends on the work-hardening parameter $\kappa$.

Supplementary to (12) and (13), we also need to specify the work-hardening coefficients $C_{KL}$ in (4). As noted earlier, a significant feature

of the material response described under 1) and 2) in Section 1 is the translation of the yield surface, and not its size. However, with reference to the phenomenon of saturation hardening examined in [4], it is the size of the yield surface which is of particular interest. In order to account also for saturation hardening, it is sufficient to specify $C_{KL}$ in the same form as that in [4, Eq. $(56)_2$], by*

$$C_{KL} = \left(\frac{\kappa-\kappa_o}{\kappa_s-\kappa_o}\right)(\beta_1\gamma_{KL} + \beta_2\gamma^p_{KL})$$

$$= \left(\frac{\kappa-\kappa_o}{\kappa_s-\kappa_o}\right)(\beta\tau_{KL} + \eta\gamma^p_{KL}) \quad , \tag{14}$$

where $\beta_1, \beta_2, \kappa_o$ and $\kappa_s$ are constants and where with the use of (11) $\beta$ and $\eta$ are given by $\beta_1 = 2\mu\beta$ and $\beta_2 = \eta - 2\mu\beta$.

With the constitutive equations (11)-(14), it follows from (6) that on the yield surface the scalar functions $\Lambda$ and $\Gamma$ assume the values

$$\Lambda = 8\mu\kappa > 0 \quad (f = g = 0) \quad , \tag{15}$$

$$\Gamma = 2\hat{\alpha}\kappa + 2[1 + (\frac{\alpha_o-\alpha_s}{\kappa_o-\kappa_s})(\tau_{KL} - \frac{\hat{\alpha}}{2}\gamma^p_{KL})\gamma^p_{KL}](\hat{\beta}\tau_{MN} + \hat{\eta}\gamma^p_{MN})(\tau_{MN} - \frac{\hat{\alpha}}{2}\gamma^p_{MN})$$
$$(f = g = 0) \quad , \tag{16}$$

where the quantities $\hat{\beta}$ and $\hat{\eta}$ are given by

$$\hat{\beta} = \hat{\beta}(\kappa) = \left(\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\right)\beta \quad , \quad \hat{\eta} = \hat{\eta}(\kappa) = \left(\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\right)\eta \tag{17}$$

subject to the restrictions

$$\hat{\beta}(\kappa_s) = \hat{\eta}(\kappa_s) = 0 \quad . \tag{18}$$

In view of (7), (15) and $(6)_3$, it is easily seen that the strain-hardening

*The hardening response coefficient in [4] is denoted by $M_{KL}$ instead of $C_{KL}$ of the present paper.

is characterized by

$$\Phi = 1 - \frac{\text{[illegible]}}{\Gamma+\Lambda} = 1 - \frac{8\mu\kappa}{\Gamma+\Lambda} < 1 \quad . \tag{19}$$

## 4. Uniaxial Cyclic Loading

Consider now a homogeneous extensional deformation sustained by a uniaxial stress $s_{11} = s(t)$ say. Adopting the notations $e_{11} = e$, $e^p_{11} = e^p$, as in [4,5] in matrix notation we write

$$\|\tau_{KL}\| = \frac{1}{3} s \|b_{KL}\| \quad , \quad \|\gamma^p_{KL}\| = \frac{1}{2} e^p \|b_{KL}\| \ , \tag{20}$$

where the constant matrix $\|b_{KL}\|$ defined by

$$\|b_{KL}\| = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \tag{21}$$

is introduced for convenience. For the uniaxial homogeneous deformation under discussion, each of the two variables e and s is represented as a line in the s-e plane (corresponding to the six-dimensional strain space and stress space). The yield surfaces are now points on these lines and the elastic range (corresponding to the interior regions of the yield surfaces) is a line segment between these points. The midpoint of the line segment corresponds to the center of the yield surface and the length of this line segment corresponds to the size of the yield surface whose interior represents the extent of the elastic region.

The loading function (12) now assumes the simplified form

$$f = \frac{2}{3} (s - \frac{3}{4} \hat{\alpha}(\kappa) e^p)^2 - \kappa \tag{22}$$

and the constitutive equations (11) and $(4)_{1,2}$ with (14)* reduce to

$$s = E(e - e^p) \tag{23}$$

*We have used the form $(14)_2$ rather that $(14)_1$ simply to make the form of the resulting equations more compact.

and

$$\dot{e}^p = \frac{\frac{4}{3}E(s - \frac{3}{4}\hat{\alpha}e^p)\dot{e}}{(\frac{4}{3}E + \hat{\alpha})(s - \frac{3}{4}\hat{\alpha}e^p) + [1 + (\frac{\alpha_o - \alpha_s}{\kappa_o - \kappa_s})(s - \frac{3}{4}\hat{\alpha}e^p)e^p](\hat{\beta}s + \frac{3}{2}\hat{\eta}e^p)} \tag{24}$$

$$\dot{\kappa} = (\hat{\beta}(\kappa)s + \frac{3}{2}\hat{\eta}(\kappa)e^p)\dot{e}^p \quad , \tag{25}$$

where E is Young's modulus. In the uniaxial case under discussion, $\Lambda$ is still given by (15) but on the yield surface $\Gamma$ and $\Phi$ assume the values

$$\Gamma_1 = 2\hat{\alpha}\kappa + \frac{4}{3}[1 + (\frac{\alpha_o - \alpha_s}{\kappa_o - \kappa_s})(s - \frac{3}{4}\hat{\alpha}e^p)e^p](\hat{\beta}s + \frac{3}{2}\hat{\eta}e^p)(s - \frac{3}{4}\hat{\alpha}e^p) \tag{26}$$

and

$$\Phi = \frac{\Gamma_1}{\Gamma_1 + \Lambda} \quad . \tag{27}$$

Then, from (23)-(25), we have

$$\frac{d\kappa}{ds} = \frac{\frac{8}{3}(\hat{\beta}s + \frac{3}{2}\hat{\eta}e^p)}{\Gamma_1} \quad . \tag{28}$$

For the special case in which the coefficients $\alpha_o, \alpha_s, \hat{\eta}, \hat{\beta}$ satisfy the conditions

$$\alpha_o = \alpha_s = \alpha \text{ (say)} \quad , \quad 2\hat{\eta} + \hat{\alpha}\hat{\beta} = 0 \quad , \tag{29}$$

after adopting $(29)_{1,2}$ and making use of the condition $f = 0$ with f given by (22), $\Gamma_1$ given by (26) reduces to

$$\Gamma_{10} = 2(\alpha + \hat{\beta})\kappa \tag{30}$$

and (28) becomes

$$\frac{d\kappa}{ds} = \frac{\pm 2\hat{\beta}(\kappa)\sqrt{\frac{2}{3}\kappa}}{\alpha + \hat{\beta}(\kappa)} \quad , \tag{31}$$

where in writing the right-hand side of (31) again use has been made of the condition $f = 0$ with $f$ given by (22) and the plus and minus signs correspond to tensile and compressive values of the stress.

In view of $(6)_3$ and (7), in the present discussion $\Gamma$ can also be used to characterize the strain-hardening behavior. Keeping this in mind and the fact that $\kappa > 0$, it follows from (30) and definitions in (8) that

$$\alpha + \hat{\beta}(\kappa) \begin{cases} > 0 \text{ for hardening} \ , & \text{(a)} \\ < 0 \text{ for softening} \ , & \text{(b)} \\ = 0 \text{ for perfectly plastic} \ . & \text{(c)} \end{cases} \tag{32}$$

The results (32) hold only for the special case in which the various coefficients satisfy $(29)_{1,2}$. They are the same as those noted previously in [12] but are obtained here in a slightly different manner.

Consider now a typical uniaxial cyclic loading which, for the $n^{th}$ cycle in the s-e plane, passes through the two points -- such as $A_n$ and $D_n$ in Fig. 1 -- whose (e,s) coordinates are $({}^n e_1^p, 0)$ and $({}^n e_2^p, 0)$ and such that

$$|{}^n e_2^p| > |{}^n e_1^p| \quad , \quad \min|e^p| = |{}^n e_1^p| \quad , \quad \max|e^p| = |{}^n e_2^p| \quad , \tag{33}$$

where the notation of double vertical bars signifies absolute value of the quantity in question, and where min and max refer to the minimum and maximum values of plastic strain attained during the $n^{th}$ cycle. Let the maximum value of compressive and tensile stresses attained during the $n^{th}$ cycle be denoted respectively by ${}^n s_1$ and ${}^n s_2$, and similarly during the $n^{th}$ cycle designate the maximum and minimum values of the strain respectively by ${}^n e_2$ and ${}^n e_1$. Then, by (23) the maximum and minimum values of the plastic strain are given by

$${}^n e_2^p = {}^n e_2 - \frac{1}{E}\,{}^n s_2 \quad , \quad {}^n e_1^p = {}^n e_1 - \frac{1}{E}\,{}^n s_1 \quad . \tag{34}$$

From the condition $f = 0$ with $f$ specified by (22), the uniaxial stress $s$ on the yield surface is found to be

$$s = \frac{3}{4}\hat{\alpha}(\kappa)e^p \pm \sqrt{\frac{3}{2}\kappa} \quad , \tag{35}$$

where the plus and minus correspond to tensile and compressive values of s respectively.

Next, as in [4], we suppose that there is a limiting value $\kappa_s$ of the work-hardening parameter $\kappa$ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \kappa = \kappa_s \quad , \tag{36}$$

so that with the help of (18) and (25)

$$\lim_{t\to\infty} \dot{\kappa} = 0 \quad . \tag{37}$$

Then, from (35), (13) and (34), the limiting value of the mean stress in a cycle may be represented as*

$$\lim_{t\to\infty} s_{mean} = \lim_{n\to\infty} \frac{1}{2}({}^n s_1 + {}^n s_2) = \frac{3}{8}\alpha_s \lim_{n\to\infty}[({}^n e_1 + {}^n e_2) - \frac{1}{E}({}^n s_1 + {}^n s_2)] \quad , \tag{38}$$

or after a slight rearrangement as

$$\lim_{t\to\infty} s_{mean} = (\frac{3E\alpha_s}{8E+6\alpha_s}) \lim_{n\to\infty}({}^n e_1 + {}^n e_2) = (\frac{3E\alpha_s}{4E+3\alpha_s}) e_{mean} \quad . \tag{39}$$

where for cycling between fixed strain limits (for which the superscript n may be suppressed) we have defined

$$e_{mean} = \frac{1}{2}\lim_{n\to\infty}({}^n e_1 + {}^n e_2) \tag{40}$$

to be the mean value of the strain limits. Moreover, in view of (18) and (36), from (26), (27), (15) and (13) we obtain

$$\lim_{t\to\infty} \Phi = \frac{\alpha_s}{\alpha_s + 4\mu} < 1 \quad . \tag{41}$$

Hence, if $\alpha_s > 0$, then by (9) and (41) the material saturates to the value

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*It should be noted that for repeated cyclic loading the number of cycles is a monotonically increasing function of time so that in the present development the limits at $t \to \infty$ and as $n \to \infty$ can be interpreted to be equivalent.

$K_h$ given by the right-hand side of (41). Correspondingly, with $\alpha_s = 0$, the right-hand side of (41) vanishes and by $(9)_3$ the material saturates to perfectly plastic behavior. Since $\dot{e}^p/\dot{s} = de^p/ds$, with the help of (23) and (24) we obtain

$$\frac{de^p}{ds} = \frac{\frac{4}{3}(s - \frac{3}{4}\hat{\alpha}e^p)}{\hat{\alpha}(s - \frac{3}{4}\hat{\alpha}e^p) + [1 + (\frac{\alpha_o - \alpha_s}{\kappa_o - \kappa_s})(s - \frac{3}{4}\hat{\alpha}e^p)e^p](\hat{\beta}s + \frac{3}{2}\hat{\eta}e^p)} , \tag{42}$$

and the slope of the stress-strain curve during loading is given by

$$\frac{ds}{de} = (\frac{de}{ds})^{-1} = (\frac{1}{E} + \frac{de^p}{ds})^{-1} . \tag{43}$$

At initial yield this has the value

$$(\frac{ds}{de})_{e^p=0,\kappa=\kappa_o} = [\frac{1}{E} + \frac{4/3}{\alpha_o+\beta}]^{-1} . \tag{44}$$

In the limit as $t \to \infty$, it follows from (43) and (42) that*

$$\lim_{t\to\infty} \frac{ds}{de} = 0 \quad \text{if} \quad \alpha_s = 0 , \tag{45a}$$

$$\lim_{t\to\infty} \frac{ds}{de} = \frac{3E\alpha_s}{4E+3\alpha_s} = \frac{9\mu k K_h}{3k+\mu K_h} \quad \text{if} \quad \alpha_s > 0 , \tag{45b}$$

where in writing the second of (45b) use has been made of (41) and (9a) and the elastic constants $\mu$ and $k$ were introduced earlier in (11).

To demonstrate the predictive capability of the basic theory for the type of response described under 1) and 2) of Section 1, we consider now a uniaxial elastic-plastic state specified by

$$s = s^*_{(1)} , \quad e^p = e^{p*} \neq 0 , \quad \kappa = \kappa^* \tag{46}$$

and suppose that the material is unloaded from this state and the load is reversed until it again yields in the reverse direction at $s = s^*_{(2)}$. For fixed values of $e^p$ and $\kappa$, $s^*_{(1)}$ and $s^*_{(2)}$ correspond to the two roots of (22) given

*The condition (45a) also follows from the first of (45b) with $\alpha_s = 0$.

by (35). It follows from the sum and difference of these roots that

$$\kappa^* = \frac{1}{6}(s^*_{(1)} - s^*_{(2)})^2 \quad , \quad \hat{\alpha}(\kappa^*) = \frac{2}{3}\,\frac{s^*_{(1)} + s^*_{(2)}}{e^{p^*}} \quad . \tag{47}$$

A value of $\kappa_o$ follows from $f = 0$ at initiation of yield and $\kappa_s$ is calculated from $(47)_1$ and the experimental data from stress reversals at saturation. It follows from (39) and $(45b)_1$ that

$$\lim_{t\to\infty} \frac{ds}{de} = \frac{\lim\limits_{t\to\infty} s_{mean}}{e_{mean}} \quad , \tag{48}$$

so that a value for the quantity $\alpha_s$ may be determined either from $(45b)_1$ and the measured slope of the stress-strain curve at saturation, or from (39) and the limiting value of the mean stress in strain cycling between fixed strain limits. An elastic-plastic state between the initial yield and saturation is specified by (46). After evaluating (13) at $\kappa = \kappa^*$ and equating the result to $(47)_2$, there results

$$\alpha_o = \frac{2}{3}\left(\frac{s^*_{(1)} + s^*_{(2)}}{e^{p^*}}\right)\left(\frac{\kappa_o - \kappa_s}{\kappa^* - \kappa_s}\right) + \alpha_s\left(\frac{\kappa^* - \kappa_o}{\kappa^* - \kappa_s}\right) \quad , \tag{49}$$

where $\kappa^*$ is calculated from $(47)_1$. Next, we introduce (49) into (44) to determine $\beta$ from the expression for the slope of the stress-strain curve at initial yield. Since $\eta$ does not occur in the yield function and does not appear in either of the limiting slopes (44) or (45a,b), its value must be determined by using the general expression resulting from (42) and (43). Hence, we need to measure the slope of the stress-strain curve at several points during loading between the initial yield and saturation. The value of $\kappa$ at such a loading point may be determined from (22) and (13), or from $(47)_1$ if it is at the point of a stress reversal. This value is then used in (42) and (43) to estimate an appropriate value for $\eta$.

The above procedure is suggested as one way of determining the material constants, and is by no means unique.

For purposes of illustration, we now specify the material constants for a material similar to the 2024-T351 aluminum alloy considered in [4]. In particular, we assume the same values for $E, \kappa_o$ and $\kappa_s$ as those given in [4, Eqs. (92) and (93)]. In addition, we specify $\alpha_s$ to be zero so that the mean stress will progressively relax and tend to zero during strain cycling. Now instead of using the procedure outlined in the preceding paragraph, the coefficients $\alpha_o, \beta$ and $\eta$ were adjusted until the predicted stress-strain curve exhibits ratcheting behavior under a loading program of stress cycling between unsymmetric stress limits*. In this way, one is able to choose the material constants as

$$\frac{\alpha_o}{E} = 0.14 \quad , \quad \alpha_s = 0 \quad , \quad \frac{\beta}{E} = 0.03 \quad , \quad \frac{\eta}{E^2} = -2.1 \times 10^{-3} \quad ,$$

$$\frac{\kappa_o}{E^2} = 1.8 \times 10^{-5} \quad , \quad \frac{\kappa_s}{E^2} = 3.0 \times 10^{-5} \quad , \tag{50}$$

$$E = 69 \text{ GPa} \quad .$$

Stress-strain response curves were calculated by explicit numerical integration of (24) using the values (50) for the material coefficients. The calculated uniaxial material response to stress cycling between fixed unsymmetric stress limits is shown in Fig. 1, and the calculated response to strain cycling between two sets of fixed strain limits is shown in Fig. 2. From (15), (26), (35), (27) and the values in (50)**, the value of $\Phi$ is calculated to be 0.10 at initial yield and it tends to zero at saturation.

Finally, a brief comparison is made of the prediction of the theory with experimental results for 304 stainless steel reported by Pugh et al. [13].

*This manner of determining the coefficients $\alpha_o, \beta$ and $\eta$ is analogous to the procedure used in [4] for obtaining different coefficients.

**Also assuming a value $\nu = 0.3$ for Poisson's ratio, with $(50)_7$ the shear modulus $\mu$ is calculated to be 26.5 GPa.

Employing the procedure described earlier [see the paragraph following (45)], the material constants were determined to be

$$\frac{\alpha_o}{E} = 1.4 \times 10^{-2} \ , \quad \frac{\alpha_s}{E} = 1.7 \times 10^{-3} \ , \quad \frac{\beta}{E} = 3.2 \times 10^{-2} \ ,$$

$$\frac{\eta}{E^2} = -1.2 \times 10^{-3} \ , \quad \frac{\kappa_o}{E^2} = 5.4 \times 10^{-7} \ , \quad \frac{\kappa_s}{E^2} = 2.7 \times 10^{-6} \ , \tag{51}$$

$$E = 123 \text{ GPa} \ .$$

The caluclated reponse is shown in Fig. 3 along with the experimental curves for the first three cycles from [13], as well as the results calculated previously by Caulk and Naghdi [4]. The only noteworthy difference between the present calculations and those in [4] is that the stress level at saturation matches the experimental data even more closely in the present calculations. From (15), (26), (35) and (27), (9), (41) and the values in* (51), the value of $\Phi$ is calculated to be 0.029 at initial yield, and it tends to the value $K_h$ =0.0053 at saturation.

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*As in [5] assuming a value $\nu = 0.3$ for Poisson's ratio, $\mu$ is calculated to be 47.3 GPa.

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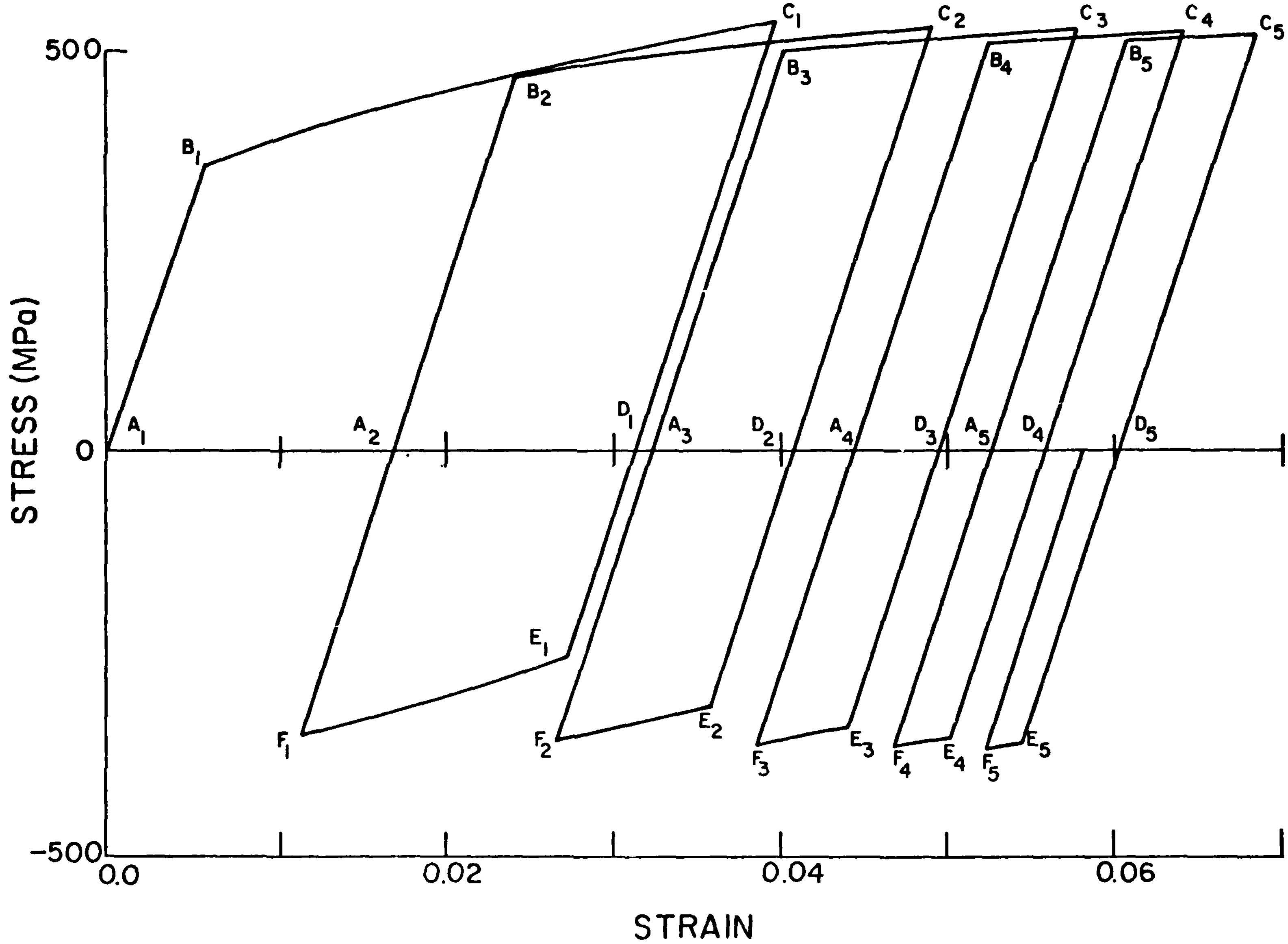
Captions for Figures

Fig. 1: Calculated uniaxial stress-strain response in s-e plane to cyclic loading between fixed values of stress in tension and compression (not necessarily equal in magnitude) resulting in ratcheting of strain. The material constants used in the calculation are given by Eq. (50).

Fig. 2: Calculated uniaxial stress-strain response in s-e plane showing progressive relaxation to zero of the mean stress caused by strain cycling between any two fixed values of strain. The material constants used in the calculation are given by Eq. (50).

Fig. 3: Comparison of theoretical cyclic stress-strain behavior for 304 stainless steel with the experimental data of reference [13]. The theoretical stress-strain curves (——) are calculated using the material constants in (51); the theoretical curves (—•—•—) are those calculated previously in [4]; and comparison with the experimental data (— — —) is shown for the first three strain cycles only, since curves for additional cycles would crowd the figure.



STRESS (MPa)
500
0
-500
0.0
0.02
0.04
0.06
STRAIN
$A_1$
$B_1$
$A_2$
$B_2$
$C_1$
$D_1$
$E_1$
$F_1$
$A_3$
$B_3$
$C_2$
$D_2$
$E_2$
$F_2$
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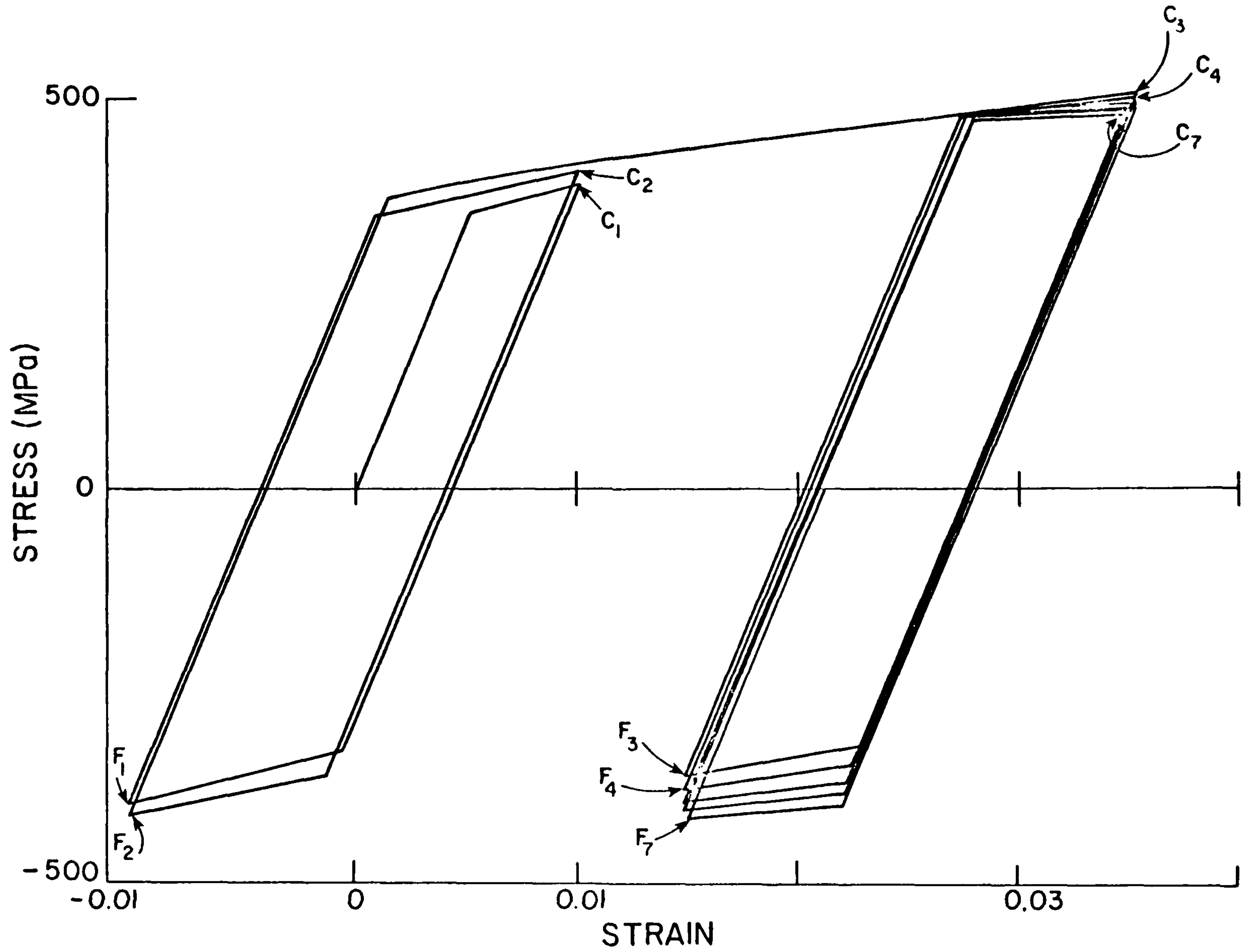
$C_5$
$D_5$
$E_5$
$F_5$


Fig. 1



500
0
-500
STRESS (MPa)
-0.01
0
0.01
0.03
STRAIN
C1
C2
C3
C4
C7
F1
F2
F3
F4
F7


Fig. 2

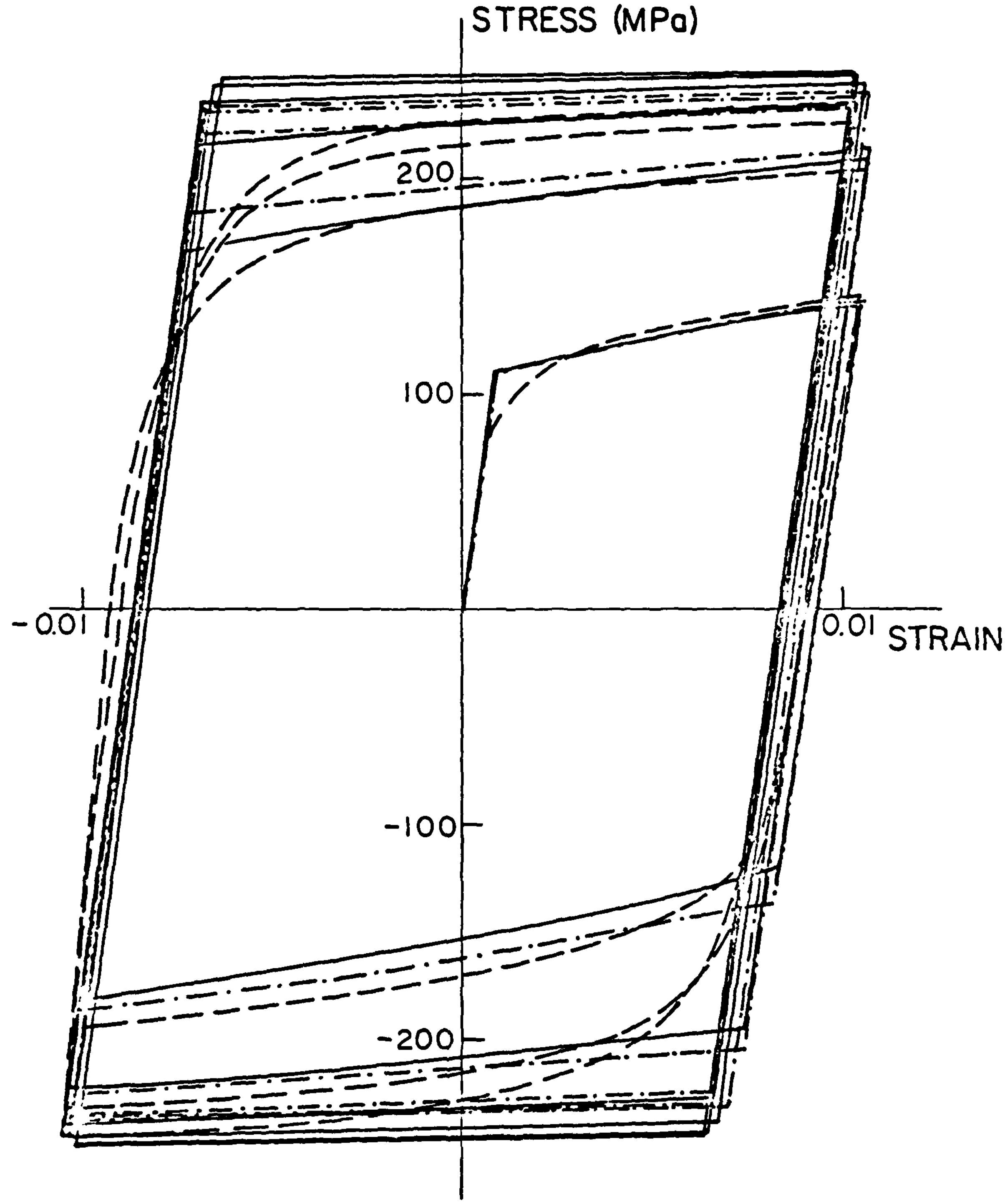



Fig. 3

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